Assessment of Time Domain Beam Propagation Methods using Padé Approximants for Optical Waveguide Analyses

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Abstract — The finite difference method (FDTD) is compared with a recently developed time domain beam propagation method (TD-BPM) using Padé approximants. Because of the slowly varying envelope a time step can be chosen, which is larger than in the case of the FDTD. In contrast the computation time for TD-BPMs will be affected by solving matrix equations including band matrices compared to simple matrix vector multiplications when using the FDTD. An investigation is applied considering both contrary effects and concluding, whether TD-BPMs can contribute to a reduction of computation time for the design of optical waveguides.

I. INTRODUCTION

The finite difference time-domain method [1] is one of the most powerful techniques for the analysis of a great variety of devices. When investigating optical waveguides FDTD methods can be inefficient as the computation time increases significantly dependent on the size of the device.

In most modern applications such as THz- and optical waveguide techniques the frequency range of interest is defined by bandwidth and centre frequency. Consequently the analysis of such devices under the prerequisite of linear and time invariant material properties can be limited to the frequency band of interest. Time domain beam propagation methods meet this specification.

Up to now different types of TD-BPMs can be distinguished, which are appropriate for the design of such devices, i. e. full-band [2], wide-band [3], and narrow band TD-BPMs [4]. All methods use an slowly varying envelope function approach. The propagation operator in time can be approximated by a well established approach based on the Padé-approximants, which is successfully used within beam propagations methods in the frequency domain [5] and recently in the time domain [6] and [7].

When comparing Padé approximant approaches the following aspects are of considerable importance: As the Padé approximant approaches allow unconditionally stable algorithms and therefore arbitrarily large time steps, which are limited by accuracy conditions, only, an advantage in decreasing the computation time may occur at first sight. Contrary to a larger time step a higher computation time has to be spent compared to the FDTD method, because matrix equations instead of simple matrix vector multiplications have to be solved.

If the time step can be chosen as high as possible, so that the computation time for FDTD methods exceeds the total computation time for TD-BPMs, then the latter method is an important alternative to commonly used FDTD methods. This contribution is aimed at this investigation. For the investigation the so called numerical dispersion caused by the finite difference approaches is a decisive criterion, as the accuracy of the solutions shall not be affected severely.

II. THEORY

For clarity the discussion is based on the scalar wave equation, the Helmholtz equation, but fundamental conclusions can still be made. Assuming a planar waveguide structure within the x-z-plane as well as a time and space dependent electrical field $\Psi = \Psi(x,z,t)$ we start with

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{n^2}{c^2}\frac{\partial^2 \Psi}{\partial x^2} . \tag{1}$$

n is the refractive index n and the velocity is described by c. The formal solution of (1) with a slowly varying complex amplitude $\psi(x,z,t)$ and a centre frequency ω_0 is given by

$$\Psi(x,z,t) = \psi(x,z,t)\, exp(j\omega_0 t). \tag{2}$$

(2) can be substituted into (1) resulting in

$$\frac{\partial^2 \psi}{\partial t^2} + 2j\frac{\partial \psi}{\partial t} = P\psi \tag{3}$$

with

$$P = \frac{c^2}{n^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) - \omega_0^2 . \tag{4}$$

The solution is given by

$$\frac{\partial \psi}{\partial t} = -j\omega_0\left(1 - \sqrt{1 - \frac{P}{\omega_0^2}}\right)\psi = [PO]\psi . \tag{5}$$

The propagation operator *[PO]* can be approximated by applying the following relation:

$$\frac{\partial \psi}{\partial t}\Big|_{n+1} = -j\frac{\dfrac{P}{2\omega_0}}{1 - \dfrac{j}{\omega_0}\dfrac{\partial \psi}{\partial t}\Big|_n} . \tag{6}$$

Hereby n denotes the order of Padé approximation.

III. DISPERSION RELATIONS

Based on the ansatz

$$\psi = exp\left(jk\Delta z\right)exp\left(j\omega_0\Delta t\right) \qquad (7)$$

dispersion relations can be derived for different orders of approximations n. A TEM-wave in z-direction will be assumed for simplicity and a classical central finite difference scheme with uniform discretization in space Δz and time Δt will be introduced. Hence, by using $x = sin(\omega\Delta t)$ and $y = cos(\omega\Delta t)$ we obtain the following dispersion relations:

$$cos(\beta\Delta z) - 1 = \left[\frac{j\omega_0}{2}y + \frac{j2x}{\Delta t}\right]\frac{j\omega_0 n^2 \Delta z^2}{c^2 y}. \qquad (8)$$

for n=0 (Padé (1,0)-approximation) and

$$cos(\beta\Delta z) - 1 = \frac{-\frac{j\omega_0}{2}y - \frac{j3}{2\Delta t}x}{\frac{jc^2}{n^2\omega_0^2\Delta t\Delta z^2}\left[x + \omega_0 y\right]}. \qquad (9)$$

for n=1 (Pade (1,1)-approximation).

IV. DISCUSSION

Due to the slowly envelope approximation numerical errors occurring at cut off frequencies are the highest within the frequency band, so for example the propagation constants applying (8) and (9) will be compared with the exact propagation constant at the lower cut off frequency. The space discretization is given by $\Delta z = \lambda_0 / 10$ and the centre frequency ω_0 corresponds to the centre wavelength of $\lambda_0 = 1{,}55\mu m$. Fig. 1 shows the propagation constants dependent on the time step for different bandwidths and Padé approximants. Explicit methods are unstable beyond $0{,}0052 \cdot 10^{-14}$ s due to the Courant stability condition [1]. As can be seen from Fig. 1, the time step can be chosen to be significantly higher than for FDTD methods when using Padé approximants.

The maximum time step can be determined for a maximum relative error of 5%. Then a ratio N between the maximum time step and a time step according to the Courant stability condition is defined. From Fig. 2 we conclude, that the ratio N increases by applying smaller widths Δz. Furthermore, Table 1 shows, that the ratio N additionally increases with smaller bandwidths.

The computation time for TD-BPMs is of the order $O(M \cdot B^2)$ and for FDTD of the order O(M). M denotes the order of the propagation matrix after discretization and B the bandwidth of the resulting propagation matrix. Finally, the bandwidth B plays an important role, as the efficiency described by the ration N is reduced by a factor of B^2. For 1-D structures we have B^2=9, as we have tridiagonal matrices, for 2-D structures, we typically have B^2=3600, for 3-D structures $B^2=10^6$, respectively. Typical values are far below $\Delta z = 0{,}02\mu m$ and 2THz. The resulting assessment leads to the conclusion, that the computing time can be considerably reduced, if narrow signal band widths are considered or a small discretization is necessary. The approach is predestined for planar waveguide structures.

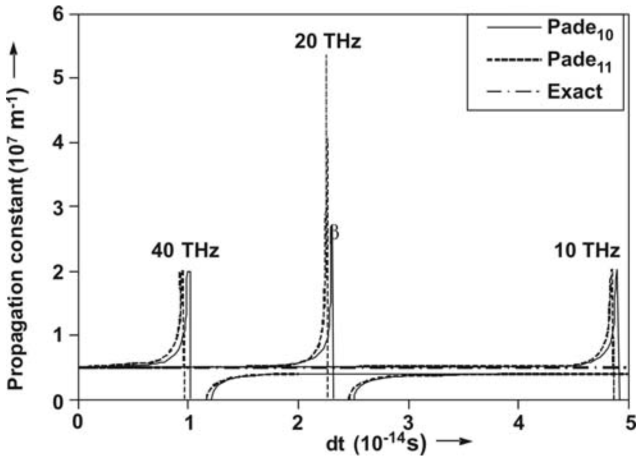

Fig. 1. Propagation constant dependent on *dt* for different bandwidths

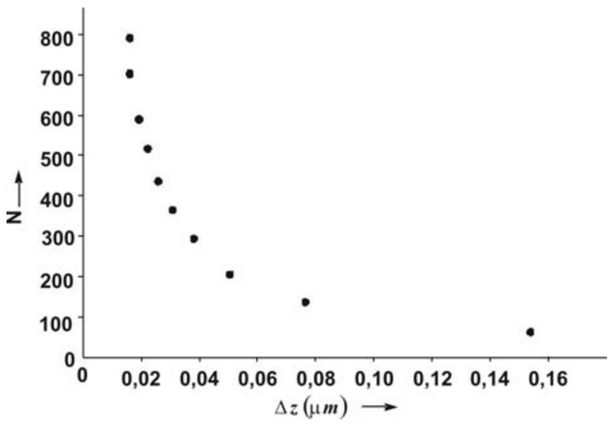

Fig. 2. N dependent on Δz for a bandwidth of 10THz

TABLE I
N for different bandwidths and discretization widths Δz

Δz /bandwidth	100 THz	40THz	10 THz	2THz
$\Delta z = 0{,}02\mu m$	≈ 19	≈ 110	≈ 525	≈ 3687
$\Delta z = 0{,}05\mu m$	≈ 5	≈ 35	≈ 220	≈ 1312
$\Delta z = 0{,}16\mu m$	≈ 1	≈ 10	≈ 73	≈ 437

V. REFERENCES

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